TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 4, Pages 1435–1452
S 0002-9947(01)02957-9
Article electronically published on December 4, 2001

WEAK AMENABILITY OF TRIANGULAR BANACH ALGEBRAS

B. E. FORREST AND L. W. MARCOUX

Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras, and let $\mathcal{M}$ be a Banach $\mathcal{A},\mathcal{B}$-module. Then $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix}$ becomes a *triangular Banach algebra* when equipped with the Banach space norm $\| \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \| = \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}$. A Banach algebra $\mathcal{T}$ is said to be *n-weakly amenable* if all derivations from $\mathcal{T}$ into its n^{th} dual space $\mathcal{T}^{(n)}$ are inner. In this paper we investigate Arens regularity and n-weak amenability of a triangular Banach algebra $\mathcal{T}$ in relation to that of the algebras $\mathcal{A}$, $\mathcal{B}$ and their action on the module $\mathcal{M}$.

1. Introduction

In general, one can obtain a good deal of information about the structure of a Banach algebra $\mathcal{A}$ by studying the various Hochschild cohomology groups $H^n(\mathcal{A}, \mathcal{X})$ of $\mathcal{A}$ with coefficients in a Banach $\mathcal{A}$-bimodule $\mathcal{X}$. For example, the study of the first cohomology group $H^1(\mathcal{A}, \mathcal{X})$ is essentially the study of inner (vs. outer) derivations from $\mathcal{A}$ into $\mathcal{X}$, while if $\mathcal{X}$ is finite-dimensional, the study of $H^2(\mathcal{A}, \mathcal{X})$ is related to that of strong splittings of extensions of $\mathcal{A}$ by $\mathcal{X}$ [1]. For this reason, it is perhaps discouraging to know that explicit calculations of cohomology groups tends to be rather difficult, and results often focus on whether a given group $H^n(\mathcal{A}, \mathcal{X})$ is or is not trivial.

In [7], motivated by work of Gilfeather and Smith [8] in their study of *joins* of operator algebras, we began an investigation of derivations of *triangular Banach algebras*; these are algebras of the form

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix},$$

where $\mathcal{A}$ and $\mathcal{B}$ are themselves Banach algebras and $\mathcal{M}$ is a Banach $\mathcal{A},\mathcal{B}$-module. In that paper a technique was developed which, in many cases, allows one to explicitly compute $H^1(\mathcal{T}, \mathcal{T})$ for specific choices of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$.

In this paper, we show that the first cohomology group for these algebras with coefficients in their n^{th} dual spaces $\mathcal{T}^{(n)}$ is also often tractable. In particular, we will develop criteria for deciding when $H^1(\mathcal{T}, \mathcal{T}^{(n)}) = 0$. This notion, referred to as n-weak amenability and defined by Dales, Ghahramani and Grønbæk in [5], splits along two distinct lines in the setting of triangular Banach algebras. Indeed, when n is odd, the (first) cohomology group of $\mathcal{T}$ depends only upon that of $\mathcal{A}$ and $\mathcal{B}$ [see Theorem 3.7 below]. When n is even, the techniques developed in [7] can be

Received by the editors October 9, 1998 and, in revised form, July 20, 1999.
2000 *Mathematics Subject Classification.* Primary 46H25, 16E40.
Research supported in part by NSERC (Canada).

extended to this setting, where it allows us to compute $H^1(\mathcal{T}, \mathcal{T}^{(2n)})$ for many choices of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{M}$ [see Theorem 3.12].

The body of this paper is divided into three parts. In Section 2, we examine the natural actions of $\mathcal{T}$ on its dual spaces $\mathcal{T}^{(n)}$, $n \geq 1$, and characterize when $\mathcal{T}$ is Arens regular. In Section 3, we obtain our two main results concerning $H^1(\mathcal{T}, \mathcal{T}^{(n)})$, the first dealing with the case where n is odd, and the second dealing with the case where n is even. Finally, in Section 4, we apply these results to three (classes) of examples, including tensor products of a given C^*-algebra $\mathcal{A}$ by $\mathcal{T}_2$, and finite dimensional nest algebras.

2. Preliminaries

2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, and suppose $\mathcal{X}$ is a *Banach* $\mathcal{A}, \mathcal{B}$*-module*; that is, $\mathcal{X}$ is a Banach space, a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, and the actions of $\mathcal{A}$ and $\mathcal{B}$ are continuous in that

$$\|a \circ x \circ b\| \leq \|a\|_{\mathcal{A}} \; \|x\|_{\mathcal{X}} \; \|b\|_{\mathcal{B}}.$$

If $\mathcal{A}$ has a unit $e_{\mathcal{A}}$ and $\mathcal{B}$ has a unit $e_{\mathcal{B}}$, then $\mathcal{X}$ is said to be *unital* provided that $e_{\mathcal{A}} \circ x = x \circ e_{\mathcal{B}} = x$ for all $x \in \mathcal{X}$.

We can define a right action of $\mathcal{A}$ on the dual space $\mathcal{X}^*$ of $\mathcal{X}$ and a left action of $\mathcal{B}$ on $\mathcal{X}^*$ via

$$\begin{aligned} (\phi \circ a)(x) &= \phi(a \circ x), \\ (b \circ \phi)(x) &= \phi(x \circ b), \end{aligned}$$

for all $a \in \mathcal{A},\ b \in \mathcal{B},\ x \in \mathcal{X}$, and $\phi \in \mathcal{X}^*$.

Similarly, the second dual $\mathcal{X}^{(2)}$ of $\mathcal{X}$ becomes a Banach $\mathcal{A}, \mathcal{B}$-module under the actions

$$\begin{aligned} (a \circ \Phi)(\phi) &= \Phi(\phi \circ a), \\ (\Phi \circ b)(\phi) &= \Phi(b \circ \phi), \end{aligned}$$

for all $a \in \mathcal{A},\ b \in \mathcal{B},\ \phi \in \mathcal{X}^*$, and $\Phi \in \mathcal{X}^{(2)}$.

Of course, we may continue this process to higher order dual spaces of $\mathcal{X}$, with the conclusion being that $\mathcal{X}^{(2m)}$ is a Banach $\mathcal{A}, \mathcal{B}$-module and $\mathcal{X}^{(2m-1)}$ is a Banach $\mathcal{B}, \mathcal{A}$-module for all $m \geq 1$. When $\mathcal{X}$ is unital, so is each $\mathcal{X}^{(m)}$, $m \geq 1$. When $\mathcal{A} = \mathcal{B}$, we simply have that each $\mathcal{X}^{(m)}$ is a Banach $\mathcal{A}$-bimodule. A particular but important example of this is when $\mathcal{X} = \mathcal{A} = \mathcal{B}$, and the module actions are given by $a \circ x = a\,x$ and $x \circ a = x\,a$ for all $a, x \in \mathcal{A}$. The above analysis then shows that $\mathcal{A}^{(m)}$ is a Banach $\mathcal{A}$-bimodule for all $m \geq 1$.

Suppose $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. A *derivation* $\delta : \mathcal{A} \to \mathcal{X}$ is a linear map which satisfies $\delta(a\,b) = \delta(a) \circ b + a \circ \delta(b)$ for all $a, b \in \mathcal{A}$. In this paper, we shall only consider continuous derivations. The derivation δ is said to be *inner* if there exists $x \in \mathcal{X}$ such that $\delta(a) = \delta_x(a) := a \circ x - x \circ a$ for all $a \in \mathcal{A}$. Denoting the linear space of bounded derivations from $\mathcal{A}$ into $\mathcal{X}$ by $Z^1(\mathcal{A}, \mathcal{X})$ and the linear subspace of (necessarily bounded) inner derivations by $N^1(\mathcal{A}, \mathcal{X})$, we may consider the quotient space $H^1(\mathcal{A}, \mathcal{X}) = Z^1(\mathcal{A}, \mathcal{X})/N^1(\mathcal{A}, \mathcal{X})$, called the *first Hochschild cohomology group of* $\mathcal{A}$ *with coefficients in* $\mathcal{X}$.

The Banach algebra $\mathcal{A}$ is said to be *amenable* if $H^1(\mathcal{A}, \mathcal{X}^*) = 0$ for all Banach $\mathcal{A}$-bimodules $\mathcal{X}$. This notion was first defined by Johnson in [10], who showed that the group algebra $L^1(G)$ is amenable precisely when G is an amenable group. In particular, the equivalence of amenability and *nuclearity* for C^*-algebras [9] indicates that amenability can be viewed as a generalized "finiteness" condition on a Banach algebra.

Bade, Curtis and Dales [1] later defined the notion of weak amenability for a commutative Banach algebra $\mathcal{A}$. They called $\mathcal{A}$ *weakly amenable* if every bounded derivation from $\mathcal{A}$ into a symmetric Banach $\mathcal{A}$-module is inner and hence zero. They also showed that this is equivalent to $H^1(\mathcal{A}, \mathcal{A}^*) = 0$. This latter condition, that $H^1(\mathcal{A}, \mathcal{A}^*) = 0$, has been adopted as the definition of weak amenability for an arbitrary Banach algebra. That this is strictly weaker than amenability is clear, since all C^*-algebras are weakly amenable [9].

More recently, Dales, Ghahramani and Grønbæk [5] have defined the notions of n-weak amenability and permanent weak amenability for a Banach algebra $\mathcal{A}$ as follows: let n be a positive integer. Then $\mathcal{A}$ is said to be *n-weakly amenable* if $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = 0$, and $\mathcal{A}$ is *permanently weakly amenable* if it is n-weakly amenable for all $n \geq 1$. Amongst other things, they were able to show that every C^*-algebra is permanently weakly amenable ([5], Theorem 2.1).

In this note we shall consider the n-weak amenability of a class of Banach algebras we first studied in [7], and which we call *triangular Banach algebras*. They are defined in the following way: let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and suppose that $\mathcal{M}$ is an essential Banach $\mathcal{A},\mathcal{B}$-module; that is, $\mathcal{A} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{B} = \mathcal{M}$. (This is guaranteed when $\mathcal{M}$ is unital.) We define the corresponding triangular Banach algebra

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix},$$

with the sum and product being given by the usual 2×2 matrix operations and obvious internal module actions. The norm on $\mathcal{T}$ is

$$\left\| \begin{bmatrix} a & m \\ & b \end{bmatrix} \right\| := \|a\|_{\mathcal{A}} + \|m\|_{\mathcal{M}} + \|b\|_{\mathcal{B}}.$$

An important class of examples arises when $\mathcal{H}$ is a complex Hilbert space, $\mathcal{A}$ and $\mathcal{B}$ are operator subalgebras of $\mathcal{B}(\mathcal{H})$, and $\mathcal{M}$ is an essential $\mathcal{A},\mathcal{B}$-submodule of $\mathcal{B}(\mathcal{H})$. $\mathcal{T}$ is still not quite an operator subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, however, since the given norm and the operator norm do not coincide.

2.2. **Arens regularity for triangular Banach algebras.** Given a triangular Banach algebra $\mathcal{T}$ as above, we can make $\mathcal{A}^{(2)}$ and $\mathcal{B}^{(2)}$ into Banach algebras by giving them the *first Arens product* as follows:

Let $\{a_i\}, \{a_j\}$ be nets in $\mathcal{A}$ with $\Gamma_1 = w^*\text{-}\lim_i a_i$ and $\Gamma_2 = w^*\text{-}\lim_j a_j$. Then we define

$$\Gamma_1 \Box \Gamma_2 = w^*\text{-}\lim_i \lim_j a_i\, a_j.$$

Similarly, we can define

$$\Psi_1 \Box \Psi_2 = w^*\text{-}\lim_i \lim_j b_i\, b_j$$

whenever $\Psi_1 = w^*\text{-}\lim_i b_i$ and $\Psi_2 = w^*\text{-}\lim_j b_j$. Moreover, we can extend the actions of $\mathcal{A}$ on $\mathcal{M}$ and of $\mathcal{B}$ on $\mathcal{M}$ to actions of $\mathcal{A}^{(2)}$ and $\mathcal{B}^{(2)}$ on $\mathcal{M}^{(2)}$ via

$$\Gamma \Box \Pi = w^*\text{-}\lim_i \lim_j a_i\, m_j$$

and

$$\Pi \Box \Psi = w^*\text{-}\lim_j \lim_k m_j\, b_k,$$

where $\Gamma = w^*\text{-}\lim_i a_i$, $\Pi = w^*\text{-}\lim_j m_j$, and $\Psi = w^*\text{-}\lim_k b_k$. This allows us to show that the first Arens multiplication defined on $\mathcal{T}^{(2)}$ behaves in a natural way.

Let $\begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix}, \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} \in \mathcal{T}^{(2)}$. Assume also that

$$\begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix} = w^*\text{-}\lim_i \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix}$$

and

$$\begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} = w^*\text{-}\lim_j \begin{bmatrix} a_j & m_j \\ & b_j \end{bmatrix}.$$

Then it is easy to see that $\Gamma_1 = w^*\text{-}\lim_i a_i, \Gamma_2 = w^*\text{-}\lim_j a_j, \Psi_1 = w^*\text{-}\lim_i b_i, \Psi_2 = w^*\text{-}\lim_j b_j, \Pi_1 = w^*\text{-}\lim_i m_i$ and $\Pi_2 = w^*\text{-}\lim_j m_j$. Moreover,

$$\begin{aligned}
\begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix} \Box \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} &= w^*\text{-}\lim_i \lim_j \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix} \begin{bmatrix} a_j & m_j \\ & b_j \end{bmatrix} \\
&= w^*\text{-}\lim_i \lim_j \begin{bmatrix} a_i a_j & a_i m_j + m_i b_j \\ & b_i b_j \end{bmatrix} \\
&= \begin{bmatrix} w^*\text{-}\lim_i \lim_j a_i a_j & w^*\text{-}\lim_i \lim_j (a_i m_j + m_i b_j) \\ & w^*\text{-}\lim_i \lim_j b_i b_j \end{bmatrix} \\
&= \begin{bmatrix} \Gamma_1 \Box \Gamma_2 & \Gamma_1 \Box \Pi_2 + \Pi_1 \Box \Psi_2 \\ & \Psi_1 \Box \Psi_2 \end{bmatrix}.
\end{aligned}$$

Thus the first Arens product of $\mathcal{T}$ behaves just like matrix multiplication, with coordinate-level operations behaving like the first Arens product of the building blocks.

There is a second way of defining a product on the second dual of a Banach algebra, namely by interchanging the order of the limits taken in the first Arens product. The result of doing this is called the *second Arens product*.

We now set

$$\Gamma_1 \diamond \Gamma_2 = w^*\text{-}\lim_j \lim_i a_i\, a_j,$$

and similarly,

$$\begin{aligned}
\Psi_1 \diamond \Psi_2 &= w^*\text{-}\lim_j \lim_i b_i\, b_j, \\
\Gamma_1 \diamond \Pi &= w^*\text{-}\lim_j \lim_i a_i\, m_j, \\
\Pi \diamond \Psi_1 &= w^*\text{-}\lim_i \lim_j m_j\, b_i.
\end{aligned}$$

The previous analysis of the first Arens product can now be extended to the second Arens product to see that

$$\begin{aligned}\begin{bmatrix} \Gamma_1 & \Pi_1 \\ & \Psi_1 \end{bmatrix} \diamond \begin{bmatrix} \Gamma_2 & \Pi_2 \\ & \Psi_2 \end{bmatrix} &= w^*\text{-}\lim_j \lim_i \begin{bmatrix} a_i & m_i \\ & b_i \end{bmatrix} \begin{bmatrix} a_j & m_j \\ & b_j \end{bmatrix} \\ &= w^*\text{-}\lim_j \lim_i \begin{bmatrix} a_i a_j & a_i m_j + m_i b_j \\ & b_i b_j \end{bmatrix} \\ &= \begin{bmatrix} w^*\text{-}\lim_j \lim_i a_i a_j & w^*\text{-}\lim_j \lim_i (a_i m_j + m_i b_j) \\ & w^*\text{-}\lim_j \lim_i b_i b_j \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_1 \diamond \Gamma_2 & \Gamma_1 \diamond \Pi_2 + \Pi_1 \diamond \Psi_2 \\ & \Psi_1 \diamond \Psi_2 \end{bmatrix}.\end{aligned}$$

In general, a Banach algebra $\mathcal{D}$ is said to be *Arens regular* provided that the first and second Arens products on $\mathcal{D}^{(2)}$ agree. Given $D, F \in \mathcal{D}^{(2)}$, it is known that $D \square F = D \diamond F$ whenever either D or F lies in the image of $\mathcal{D}$ under the canonical embedding of $\mathcal{D}$ into $\mathcal{D}^{(2)}$ ([13], Theorem 1.4.2) *A fortiori*, when this is the case, the Arens product also coincides with the natural dual module action of $\mathcal{D}$ on $\mathcal{D}^{(2)}$ defined at the beginning of Section 2.1. To see this, suppose for instance that D, $F \in \mathcal{D}^{(2)}$ with D being the image of some $d \in \mathcal{D}$ under the canonical embedding. Suppose $F = w^*\text{-}\lim_j f_j$ and consider $\rho \in \mathcal{D}^*$. Then $(D \square F) = w^*\text{-}\lim_j d\, f_j$ and so

$$(D \square F)(\rho) = \lim_j \rho(d\, f_j).$$

On the other hand, we have

$$\begin{aligned}(D \circ F)(\rho) &= F(\rho \circ d) \\ &= \lim_j (\rho \circ d)(f_j) \\ &= \lim_j \rho(d\, f_j) \\ &= (D \square F)(\rho).\end{aligned}$$

The case where F arises from the image of some $f \in \mathcal{D}$ under the canonical embedding and $D \in \mathcal{D}^{(2)}$ is arbitrary is handled in a similar fashion.

Returning to the setting of triangular Banach algebras, the same observations carry over to the case of the two $\mathcal{A}^{(2)}, \mathcal{B}^{(2)}$-module actions $\diamond$ and $\square$ which we have defined on $\mathcal{M}^{(2)}$. More precisely, $\Gamma \diamond \Pi = \Gamma \square \Pi$ if either Γ is the image of some $a \in \mathcal{A}$ under the canonical embedding of $\mathcal{A}$ into $\mathcal{A}^{(2)}$, or if Π is the image of some $m \in \mathcal{M}$ under the canonical embedding of $\mathcal{M}$ into its second dual. (A similar statement holds for the right action of $\mathcal{B}^{(2)}$ upon $\mathcal{M}^{(2)}$.) Again, when at least one term is the image of an element in $\mathcal{A}$ (or $\mathcal{M}$, or $\mathcal{B}$), these two actions $\square$ and $\diamond$ agree with the dual module action $\circ$ defined above. We shall return to this later.

Let us say that the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ act *regularly* on $\mathcal{M}$ if for every $\Gamma \in \mathcal{A}^{(2)}$, $\Pi \in \mathcal{M}^{(2)}$ and $\Psi \in \mathcal{B}^{(2)}$ we have

$$\Gamma \square \Pi = \Gamma \diamond \Pi \quad \text{and} \quad \Pi \square \Psi = \Pi \diamond \Psi.$$

The following proposition is now immediate:

2.3. Proposition. *A triangular Banach algebra* $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ *is Arens regular if and only if both* $\mathcal{A}$ *and* $\mathcal{B}$ *are Arens regular and* $\mathcal{A}$ *and* $\mathcal{B}$ *act regularly on* $\mathcal{M}$.

2.4. Now that we have shown that $\mathcal{T}^{(2)}$ can be made into a Banach algebra using either Arens product, we may apply the same argument to show that we can make $\mathcal{T}^{(4)}$, $\mathcal{T}^{(6)}$, and more generally $\mathcal{T}^{(2n)}$ for $n \geq 1$ into a Banach algebra by repeatedly using the first (or second) Arens product. Two interesting implications of this analysis are the following:

(i) It provides us with an action of $\mathcal{T}$ upon $\mathcal{T}^{(2n)}$ for each $n \geq 1$ via restriction of the Arens product to the canonical embedding of $\mathcal{T}$ into $\mathcal{T}^{(2n)}$. As we have seen, this action agrees with the dual module action defined in Section 2.1.

(ii) The action of $\mathcal{T}$ upon $\mathcal{T}^{(2n)}$ referred to in the previous paragraph in turn defines "actions" of $\mathcal{A}^{(2n)}$ and $\mathcal{B}^{(2n)}$ on $\mathcal{M}$ *taking values in* $\mathcal{M}^{(2n)}$ via the formulae

$$A \circ m = A \diamond \tilde{m} \quad \text{and} \quad m \circ B = \tilde{m} \diamond B,$$

where $A \in \mathcal{A}^{(2n)}$, $B \in \mathcal{B}^{(2n)}$, and $\tilde{m}$ denotes the (canonical) embedding of an element $m \in \mathcal{M}$ into $\mathcal{M}^{(2n)}$. We shall use this in our study of derivations of $\mathcal{T}$ into $\mathcal{T}^{(2n)}$, $n \geq 1$.

2.5. **Iterated duals - the actions of $\mathcal{T}$ upon $\mathcal{T}^{(m)}$.** Since, as a Banach space, $\mathcal{T}$ is isomorphic to the ℓ^1-direct sum of $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{B}$, it is clear that $\mathcal{T}^{(2m-1)} \simeq \mathcal{A}^{(2m-1)} \oplus_1 \mathcal{M}^{(2m-1)} \oplus_1 \mathcal{B}^{(2m-1)}$, while $\mathcal{T}^{(2m)} \simeq \mathcal{A}^{(2m)} \oplus_\infty \mathcal{M}^{(2m)} \oplus_\infty \mathcal{B}^{(2m)}$ for each $m \geq 1$. Let us now consider the action of $\mathcal{T}$ upon $\mathcal{T}^{(m)}$, $m \geq 1$. To reduce notational clutter, we shall denote the module actions of $\mathcal{A}$ and $\mathcal{B}$ upon $\mathcal{M}$ simply by juxtaposition.

The case $m = 1$. Suppose $t = \begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$ and $\tau = \begin{bmatrix} \alpha & \mu \\ & \beta \end{bmatrix} \in \mathcal{T}^*$. (Although we write τ as an upper triangular matrix, this is simply for convenience. The action of $\mathcal{T}^*$ upon $\mathcal{T}$ is given by $\tau(t) = \alpha(a) + \mu(m) + \beta(b)$.)

Let $w = \begin{bmatrix} x & y \\ & z \end{bmatrix} \in \mathcal{T}$. From above, the action of w on τ is given by

$$\begin{aligned}
(w \circ \tau)(t) &= \tau(tw) \\
&= \tau \begin{bmatrix} ax & ay + mz \\ & bz \end{bmatrix} \\
&= \alpha(ax) + \mu(ay + mz) + \beta(bz) \\
&= (x \circ \alpha)(a) + (y \circ \mu)(a) + (z \circ \mu)(m) + (z \circ \beta)(b) \\
&= \begin{bmatrix} x \circ \alpha + y \circ \mu & z \circ \mu \\ & z \circ \beta \end{bmatrix} \begin{pmatrix} a & m \\ & b \end{pmatrix}.
\end{aligned}$$

Thus $(w \circ \tau) = \begin{bmatrix} x \circ \alpha + y \circ \mu & z \circ \mu \\ & z \circ \beta \end{bmatrix}$.

Similarly, one can check that $(\tau \circ w) = \begin{bmatrix} \alpha \circ x & \mu \circ x \\ & \mu \circ y + \beta \circ z \end{bmatrix}$.

The case $m = 2$. We can identify $\mathcal{T}^{(2)}$ with $\begin{bmatrix} \mathcal{A}^{(2)} & \mathcal{M}^{(2)} \\ & \mathcal{B}^{(2)} \end{bmatrix}$. As we have seen, the action of $\mathcal{T}$ upon $\mathcal{T}^{(2)}$ coincides with the restriction of the (first or second) Arens product on $\mathcal{T}^{(2)}$ to the image of $\mathcal{T}$ in $\mathcal{T}^{(2)}$ under the canonical embedding, and hence

$$\begin{bmatrix} x & y \\ & z \end{bmatrix} \circ \begin{bmatrix} \Gamma & \Pi \\ & \Phi \end{bmatrix} = \begin{bmatrix} x \circ \Gamma & x \circ \Pi + y \circ \Phi \\ & z \circ \Phi \end{bmatrix}$$

and

$$\begin{bmatrix} \Gamma & \Pi \\ & \Phi \end{bmatrix} \circ \begin{bmatrix} x & y \\ & z \end{bmatrix} = \begin{bmatrix} \Gamma \circ x & \Gamma \circ y + \Pi \circ z \\ & \Phi \circ z \end{bmatrix}.$$

Again, we observe that these equations coincide with the standard matrix "multiplications" of w and T, with multiplication at the coordinate level corresponding to the appropriate module action.

THE CASE $m \geq 3$. It is not hard to see that the action of $\mathcal{T}$ on $\mathcal{T}^{(3)}$ agrees with the dual action of $\mathcal{T}^{(2)}$ on $\mathcal{T}^{(3)}$ when we restrict to the image of $\mathcal{T}$ in $\mathcal{T}^{(2)}$ under the canonical embedding. Indeed, note that if $\begin{bmatrix} \theta & \eta \\ & \omega \end{bmatrix} \in \mathcal{T}^{(3)}$, $\begin{bmatrix} \Gamma & \Pi \\ & \Psi \end{bmatrix} \in \mathcal{T}^{(2)}$, and $\begin{bmatrix} a & m \\ & b \end{bmatrix} \in \mathcal{T}$, then

$$\begin{aligned} \begin{bmatrix} \theta & \eta \\ & \omega \end{bmatrix} \circ \begin{bmatrix} a & m \\ & b \end{bmatrix} \begin{pmatrix} \Gamma & \Pi \\ & \Psi \end{pmatrix} &= \begin{bmatrix} \theta & \eta \\ & \omega \end{bmatrix} \left(\begin{bmatrix} a & m \\ & b \end{bmatrix} \circ \begin{bmatrix} \Gamma & \Pi \\ & \Psi \end{bmatrix} \right) \\ &= \begin{bmatrix} \theta & \eta \\ & \omega \end{bmatrix} \left(\begin{bmatrix} a & m \\ & b \end{bmatrix} \diamond \begin{bmatrix} \Gamma & \Pi \\ & \Psi \end{bmatrix} \right), \end{aligned}$$

where $\circ$ denotes the appropriate dual action and $\diamond$ denotes the Arens product (recall that both Arens products agree since one of the terms is the image of an element of $\mathcal{T}$ under the canonical embedding). More generally, it follows that the computation of the action of $\mathcal{T}$ upon $\mathcal{T}^*$ carries over *mutatis mutandis* to the action of $\mathcal{T}$ upon $\mathcal{T}^{(2m-1)}$, $m \geq 2$. Similarly, the action of $\mathcal{T}$ upon $\mathcal{T}^{(2m)}$, $m \geq 2$, also satisfies the corresponding formulae for the action of $\mathcal{T}$ upon $\mathcal{T}^{(2)}$, and hence "looks like" standard matrix multiplication, as we saw in Section 2.4.

3. Derivations and n-weak amenability

3.1. We now wish to consider the question of n-weak amenability of triangular Banach algebras $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$. From this point on, we shall assume that the corner algebras $\mathcal{A}$ and $\mathcal{B}$ are unital and that $\mathcal{M}$ is a unital Banach $\mathcal{A},\mathcal{B}$-module. Although this is not crucial for weak amenability (i.e. when $n = 1$), it guarantees that $\mathcal{M}^{(n)}$ is an essential module when $n \geq 2$. As we shall see, there are two distinct phenomena which occur, depending upon whether n is even or odd. We begin by considering the case where n is odd, and in fact, first considering the case where $n = 1$.

Weak amenability. The proof of the following lemma involves nothing more than routine calculations which are left to the reader.

3.2. Lemma. *Let $D : \mathcal{T} \to \mathcal{T}^*$ be a continuous derivation. Then there exist continuous derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^*$, $\delta_4 : \mathcal{B} \to \mathcal{B}^*$, and an element $\gamma_\delta \in \mathcal{M}^*$ such that*

(i) $D \begin{bmatrix} x & 0 \\ & 0 \end{bmatrix} = \begin{bmatrix} \delta_1(x) & \gamma_\delta \circ x \\ & 0 \end{bmatrix}$ *for all $x \in \mathcal{A}$,*

(ii) $D \begin{bmatrix} 0 & 0 \\ & z \end{bmatrix} = \begin{bmatrix} 0 & -z \circ \gamma_\delta \\ & \delta_4(z) \end{bmatrix}$ *for all $z \in \mathcal{B}$,*

(iii) $D \begin{bmatrix} 0 & y \\ & 0 \end{bmatrix} = \begin{bmatrix} -y \circ \gamma_\delta & 0 \\ & \gamma_\delta \circ y \end{bmatrix}$ *for all $y \in \mathcal{M}$.*

3.3. Lemma. *Suppose $\delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^*$ is a continuous derivation. Then*

$$\begin{array}{cccc} D_{\delta_{\mathcal{A}}} : & \mathcal{T} & \to & \mathcal{T}^*, \\ & \begin{bmatrix} a & m \\ & b \end{bmatrix} & \mapsto & \begin{bmatrix} \delta_{\mathcal{A}}(a) & 0 \\ & 0 \end{bmatrix}, \end{array}$$

is also a continuous derivation. Furthermore, $\delta_{\mathcal{A}}$ is inner if and only if $D_{\delta_{\mathcal{A}}}$ is inner.

Similarly, suppose $\delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^*$ is a continuous derivation. Then

$$\begin{array}{cccc} D_{\delta_{\mathcal{B}}} : & \mathcal{T} & \to & \mathcal{T}^*, \\ & \begin{bmatrix} a & m \\ & b \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 0 \\ & \delta_{\mathcal{B}}(b) \end{bmatrix}, \end{array}$$

is a continuous derivation, and it is inner precisely when $\delta_{\mathcal{B}}$ is inner.

Proof. That $D_{\delta_{\mathcal{A}}}$ and $D_{\delta_{\mathcal{B}}}$ are derivations are again elementary calculations which we leave to the reader.

Suppose $\delta_{\mathcal{A}}$ is inner. Then there exists $\alpha \in \mathcal{A}^*$ such that $\delta_{\mathcal{A}}(a) = \delta_\alpha(a) := a \circ \alpha - \alpha \circ a$.

Consider $\tau = \begin{bmatrix} \alpha & 0 \\ & 0 \end{bmatrix} \in \mathcal{T}^*$. Then

$$\begin{aligned} D_\tau \begin{bmatrix} a & m \\ & b \end{bmatrix} &= \begin{bmatrix} a & m \\ & b \end{bmatrix} \circ \begin{bmatrix} \alpha & 0 \\ & 0 \end{bmatrix} - \begin{bmatrix} \alpha & 0 \\ & 0 \end{bmatrix} \circ \begin{bmatrix} a & m \\ & b \end{bmatrix} \\ &= \begin{bmatrix} a \circ \alpha - \alpha \circ a & 0 \\ & 0 \end{bmatrix} \\ &= \begin{bmatrix} \delta_\alpha(a) & 0 \\ & 0 \end{bmatrix} \\ &= D_{\delta_{\mathcal{A}}} \begin{bmatrix} a & m \\ & b \end{bmatrix}, \end{aligned}$$

and so $D_{\delta_{\mathcal{A}}}$ is inner.

Conversely, suppose $D_{\delta_{\mathcal{A}}}$ is inner. Then there exists $\tau = \begin{bmatrix} \alpha & \mu \\ & \beta \end{bmatrix} \in \mathcal{T}^*$ such that

$$\begin{aligned} D_{\delta_{\mathcal{A}}} \begin{bmatrix} a & m \\ & b \end{bmatrix} &= \begin{bmatrix} a & m \\ & b \end{bmatrix} \circ \begin{bmatrix} \alpha & \mu \\ & \beta \end{bmatrix} - \begin{bmatrix} \alpha & \mu \\ & \beta \end{bmatrix} \circ \begin{bmatrix} a & m \\ & b \end{bmatrix} \\ &= \begin{bmatrix} a \circ \alpha + m \circ \mu - \alpha \circ a & b \circ \mu - \mu \circ a \\ & b \circ \beta - \mu \circ m - \beta \circ b \end{bmatrix}. \end{aligned}$$

But

$$D_{\delta_{\mathcal{A}}} \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \delta_{\mathcal{A}}(a) & 0 \\ & 0 \end{bmatrix}.$$

It follows that $\delta_{\mathcal{A}}(a) = a \circ \alpha - \alpha \circ a + m \circ \mu$ for all $a \in \mathcal{A}$, $m \in \mathcal{M}$. We may assume without loss of generality that $m = 0$. Then for any $a \in \mathcal{A}$, we have

$$\delta_{\mathcal{A}}(a) = a \circ \alpha - \alpha \circ a,$$

and so $\delta_{\mathcal{A}} = \delta_\alpha$ is inner, as claimed. The proofs for $\delta_{\mathcal{B}}$ and $D_{\delta_{\mathcal{B}}}$ are similar. $\square$

We mentioned in the above proof that the calculation that shows that derivations from the corner algebra $\mathcal{A}$ into its dual space lift to derivations from $\mathcal{T}$ into $\mathcal{T}^*$ is elementary. It is worth pointing out that the conclusion is *false* if we consider lifting derivations from $\mathcal{A}$ into $\mathcal{A}$ to derivations from $\mathcal{T}$ into $\mathcal{T}$. See [7], Example 3.6 for a counterexample.

3.4. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}$-module. Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ be the corresponding triangular Banach algebra. Then*

$$H^1(\mathcal{T},\mathcal{T}^*) \simeq H^1(\mathcal{A},\mathcal{A}^*) \oplus H^1(\mathcal{B},\mathcal{B}^*).$$

Proof. Let $\delta : \mathcal{T} \to \mathcal{T}^*$ be a (continuous) derivation. From Lemma 3.2 above, we obtain derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^*$ and $\delta_4 : \mathcal{B} \to \mathcal{B}^*$, as well as an element $\gamma_\delta \in \mathcal{M}^*$ such that

$$\delta \begin{bmatrix} x & y \\ & z \end{bmatrix} = \begin{bmatrix} \delta_1(x) - y \circ \gamma_\delta & \gamma_\delta \circ x - z \circ \gamma_\delta \\ & \gamma_\delta \circ y + \delta_4(z) \end{bmatrix}.$$

Consider the linear map

$$\begin{array}{rccc} \kappa : & Z^1(\mathcal{T},\mathcal{T}^*) & \to & H^1(\mathcal{A},\mathcal{A}^*) \oplus H^1(\mathcal{B},\mathcal{B}^*), \\ & \delta & \mapsto & (\delta_1 + N^1(\mathcal{A},\mathcal{A}^*), \delta_4 + N^1(\mathcal{B},\mathcal{B}^*)). \end{array}$$

Clearly κ is linear.

We claim that κ is surjective. Indeed, given $\delta_{\mathcal{A}} \in Z^1(\mathcal{A},\mathcal{A}^*)$ and $\delta_{\mathcal{B}} \in Z^1(\mathcal{B},\mathcal{B}^*)$, Lemma 3.3 shows that the lifted maps $D_{\delta_{\mathcal{A}}}$ and $D_{\delta_{\mathcal{B}}}$ both lie in $Z^1(\mathcal{T},\mathcal{T}^*)$, and thus $D = D_{\delta_{\mathcal{A}}} + D_{\delta_{\mathcal{B}}} \in Z^1(\mathcal{T},\mathcal{T}^*)$. But then

$$\begin{aligned} \kappa(D) &= \kappa(D_{\delta_{\mathcal{A}}}) + \kappa(D_{\delta_{\mathcal{B}}}) \\ &= (\delta_{\mathcal{A}} + N^1(\mathcal{A},\mathcal{A}^*), 0) + (0, \delta_{\mathcal{B}} + N^1(\mathcal{B},\mathcal{B}^*)) \\ &= (\delta_{\mathcal{A}} + N^1(\mathcal{A},\mathcal{A}^*), \delta_{\mathcal{B}} + N^1(\mathcal{B},\mathcal{B}^*)). \end{aligned}$$

Thus κ is onto, as claimed.

Next, suppose $\delta \in \ker \kappa$. Then $\delta_1 \in N^1(\mathcal{A},\mathcal{A}^*)$ and $\delta_4 \in N^1(\mathcal{B},\mathcal{B}^*)$. By Lemma 3.3, $D_{\delta_1}, D_{\delta_4} \in N^1(\mathcal{T},\mathcal{T}^*)$, and hence $D = D_{\delta_1} + D_{\delta_4} \in N^1(\mathcal{T},\mathcal{T}^*)$. But $\delta - D = \delta_{\begin{bmatrix} 0 & \gamma_\delta \\ & 0 \end{bmatrix}} \in N^1(\mathcal{T},\mathcal{T}^*)$, and so δ is an inner derivation. That $\delta \in N^1(\mathcal{T},\mathcal{T}^*)$ implies δ_1, δ_4 are also inner is again Lemma 3.3. Thus $\ker \kappa = N^1(\mathcal{T},\mathcal{T}^*)$.

Elementary linear algebra theory now implies that

$$H^1(\mathcal{A},\mathcal{A}^*) \oplus H^1(\mathcal{B},\mathcal{B}^*) = \operatorname{ran} \kappa \simeq Z^1(\mathcal{T},\mathcal{T}^*)/\ker \kappa = H^1(\mathcal{T},\mathcal{T}^*).$$

□

3.5. Corollary. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A},\mathcal{B}$-module. Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ be the corresponding triangular Banach algebra. Then $\mathcal{T}$ is weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable.*

3.6. Recall that the action of $\mathcal{T}$ on $\mathcal{T}^{(3)}$ is a restriction of the action of $\mathcal{T}^{(2)}$ on $\mathcal{T}^{(3)}$. From this we can again determine the behaviour of a derivation $D : \mathcal{T} \to \mathcal{T}^{(3)}$. As before,

$$\begin{bmatrix} \theta & \eta \\ & \omega \end{bmatrix} \circ \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} \theta \circ a & \eta \circ a \\ & \eta \circ m + \omega \circ b \end{bmatrix}$$

and

$$\begin{bmatrix} a & m \\ & b \end{bmatrix} \circ \begin{bmatrix} \theta & \eta \\ & \omega \end{bmatrix} = \begin{bmatrix} a \circ \theta + m \circ \eta & b \circ \eta \\ & b \circ \omega \end{bmatrix}.$$

By repeating exactly the same calculations as for derivations δ from $\mathcal{T}$ into $\mathcal{T}^*$, we find that there exist derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^{(3)}$ and $\delta_4 : \mathcal{B} \to \mathcal{B}^{(3)}$ as well as an element $\gamma_\delta \in \mathcal{M}^{(3)}$ (arising as the $(1,2)$-coordinate of the image of $\delta\left(\begin{bmatrix} e_{\mathcal{A}} & 0 \\ & 0 \end{bmatrix}\right))$ such that

$$\delta(\begin{bmatrix} a & m \\ & b \end{bmatrix}) = \begin{bmatrix} \delta_1(a) & 0 \\ & \delta_4(b) \end{bmatrix} + \delta_{\begin{bmatrix} 0 & \gamma_\delta \\ & 0 \end{bmatrix}}(\begin{bmatrix} a & m \\ & b \end{bmatrix}).$$

This shows that $H^1(\mathcal{T}, \mathcal{T}^{(3)}) \simeq H^1(\mathcal{A}, \mathcal{A}^{(3)}) \oplus H^1(\mathcal{B}, \mathcal{B}^{(3)})$, and that, in particular, $\mathcal{T}$ is 3-weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are. In fact, this argument extends to any dual of the form $\mathcal{T}^{(2n-1)}$ with $n \geq 1$.

We therefore have:

3.7. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A}, \mathcal{B}$-module. Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ be the corresponding triangular Banach algebra. Suppose n is a positive integer. Then*

$$H^1(\mathcal{T}, \mathcal{T}^{(2n-1)}) \simeq H^1(\mathcal{A}, \mathcal{A}^{(2n-1)}) \oplus H^1(\mathcal{B}, \mathcal{B}^{(2n-1)}).$$

It follows that $\mathcal{T}$ is $(2n-1)$- weakly amenable if and only if both $\mathcal{A}$ and $\mathcal{B}$ are.

3.8. $(2n)$**-weak amenability.** The study of derivations from $\mathcal{T}$ into $\mathcal{T}^{(2n)}$ differs radically from that of derivations into $\mathcal{T}^{(2n-1)}$, $n \geq 1$. While the $(2n-1)$-weak amenability of $\mathcal{T}$ depends only upon the $(2n-1)$-weak amenability of $\mathcal{A}$ and $\mathcal{B}$, the situation for $(2n)$-weak amenability more closely resembles the situation explored in [7] for derivations from $\mathcal{T}$ into itself. Indeed, given that the action of $\mathcal{T}$ upon $\mathcal{T}^{(2n)}$ mimics that of matrix multiplication, many of the calculations from that paper formally carry over intact to prove an analogue of [7], Thm. 2.8. For the sake of completeness, we shall include a detailed outline of the proof here, elaborating on the differences which occur as a result of having changed the codomain of the derivations.

3.9. Proposition. *Let n be a positive integer and suppose $\delta : \mathcal{T} \to \mathcal{T}^{(2n)}$ is a continuous derivation. Then there exist $\gamma_\delta \in \mathcal{M}^{(2n)}$, and continuous derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^{(2n)}$, $\delta_4 : \mathcal{B} \to \mathcal{B}^{(2n)}$ and a continuous map $\rho : \mathcal{M} \to \mathcal{M}^{(2n)}$ which satisfy:*

(i) $\delta \begin{bmatrix} a & 0 \\ & 0 \end{bmatrix} = \begin{bmatrix} \delta_1(a) & a \circ \gamma_\delta \\ & 0 \end{bmatrix}$;

(ii) $\delta \begin{bmatrix} 0 & 0 \\ & b \end{bmatrix} = \begin{bmatrix} 0 & -\gamma_\delta \circ b \\ & \delta_4(b) \end{bmatrix}$;

(iii) $\delta \begin{bmatrix} 0 & m \\ & 0 \end{bmatrix} = \begin{bmatrix} 0 & \rho(m) \\ & 0 \end{bmatrix}$;

(iv) $\rho(a \circ m) = \delta_1(a) \circ m + a \circ \rho(m)$;

(v) $\rho(m \circ b) = \rho(m) \circ b + m \circ \delta_4(b)$.

(vi) *Futhermore, if* $\delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}^{(2n)}$ *and* $\delta_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^{(2n)}$ *are continuous derivations and* $\rho_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^{(2n)}$ *is a continuous linear map that satisfies* (iv) *and* (v), *then*

$$\Delta \begin{bmatrix} a & m \\ & b \end{bmatrix} := \begin{bmatrix} \delta_{\mathcal{A}}(a) & \rho_{\mathcal{M}}(m) \\ & \delta_{\mathcal{B}}(b) \end{bmatrix}$$

defines a continuous derivation of $\mathcal{T}$ *into* $\mathcal{T}^{(2n)}$.

Proof. The proofs of these statements rely on elementary matrix calculations and are left to the reader. □

3.10. Given a Banach $\mathcal{A}, \mathcal{B}$-module $\mathcal{M}$ as we have, let us consider maps of the form

$$\begin{array}{rccc} \rho_{x,z} : & \mathcal{M} & \to & \mathcal{M}^{(2n)}, \\ & m & \mapsto & x \circ m - m \circ z, \end{array}$$

where $x \in \mathcal{A}^{(2n)}$, $z \in \mathcal{B}^{(2n)}$. We shall denote the centralizer of $\mathcal{A}$ in $\mathcal{A}^{(2n)}$ as $Z_{\mathcal{A}}(\mathcal{A}^{(2n)}) = \{x \in \mathcal{A}^{(2n)} : x \circ a = a \circ x \text{ for all } a \in \mathcal{A}\}$, and similarly, $Z_{\mathcal{B}}(\mathcal{B}^{(2n)}) = \{z \in \mathcal{B}^{(2n)} : z \circ b = b \circ z \text{ for all } b \in \mathcal{B}\}$. In analogy with [7], we shall refer to the set

$$ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) = \{\rho_{x,z} : \mathcal{M} \to \mathcal{M}^{(2n)}; x \in Z_{\mathcal{A}}(\mathcal{A}^{(2n)}),\ z \in Z_{\mathcal{B}}(\mathcal{B}^{(2n)}\}$$

as *central Rosenblum operators on* $\mathcal{M}$ *with coefficients in* $\mathcal{M}^{(2n)}$.

We also define

$$\begin{aligned} Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) \ &= \{\phi : \mathcal{M} \to \mathcal{M}^{(2n)}; \phi(a \circ m) = a \circ \phi(m), \phi(m \circ b) \\ &\qquad = \phi(m) \circ b \text{ for all } a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B}\}. \end{aligned}$$

We then have

3.11. Lemma. (i) $ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) \subseteq Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$.

(ii) *If* $\phi \in Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$, *then the map*

$$\Delta_\phi \begin{bmatrix} a & m \\ & b \end{bmatrix} = \begin{bmatrix} 0 & \phi(m) \\ & 0 \end{bmatrix} \in Z^1(\mathcal{T}, \mathcal{T}^{(2n)}).$$

Moreover, Δ_ϕ *is inner if and only if* ϕ *is a central Rosenblum operator on* $\mathcal{M}$ *with coefficients in* $\mathcal{M}^{(2n)}$.

Proof. (i) Suppose $\rho_{x,z} \in ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$, and that $a \in \mathcal{A}$, $b \in \mathcal{B}$. Then

$$\begin{aligned} \rho_{x,z}(a \circ m) &= x \circ (a \circ m) - (a \circ m) \circ z \\ &= (x \circ a) \circ m - a \circ (m \circ z) \\ &= (a \circ x) \circ m - a \circ (m \circ z) \\ &= a \circ (x \circ m - m \circ z) \\ &= a \circ \rho_{x,z}(m). \end{aligned}$$

Similarly, $\rho_{x,z}(m \circ b) = \rho_{x,z}(m) \circ b$.

(ii) The first statement is Proposition 3.9 [(vi)] applied to the case where $\delta_{\mathcal{A}} = \delta_{\mathcal{B}} = 0$. Suppose $\phi = \rho_{x,z} \in Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$. Then

$$\begin{aligned} \delta_{\begin{bmatrix} x & 0 \\ & z \end{bmatrix}} \begin{pmatrix} a & m \\ & b \end{pmatrix} &= \begin{bmatrix} x \circ a - a \circ x & x \circ m - m \circ z \\ & z \circ b - b \circ z \end{bmatrix} \\ &= \begin{bmatrix} 0 & \phi(m) \\ & 0 \end{bmatrix} \\ &= \Delta_\phi \begin{pmatrix} a & m \\ & b \end{pmatrix}. \end{aligned}$$

Hence Δ_ϕ is inner.

Conversely, suppose Δ_ϕ is inner, say $\Delta_\phi = \delta_{\begin{bmatrix} x & y \\ & z \end{bmatrix}}$ with $\begin{bmatrix} x & y \\ & z \end{bmatrix} \in \mathcal{T}^{(2n)}$. Then

$$\begin{aligned} \delta_{\begin{bmatrix} x & y \\ & z \end{bmatrix}} \begin{pmatrix} z & m \\ & b \end{pmatrix} &= \begin{bmatrix} x \circ a - a \circ x & x \circ m - m \circ z + y \circ b - a \circ y \\ & z \circ b - b \circ z \end{bmatrix} \\ &= \begin{bmatrix} 0 & \phi(m) \\ & 0 \end{bmatrix}. \end{aligned}$$

Hence $x \in Z_{\mathcal{A}}(\mathcal{A}, \mathcal{A}^{(2n)})$ and $z \in Z_{\mathcal{B}}(\mathcal{B}, \mathcal{B}^{(2n)})$. Moreover, $\phi(m) = x \circ m - m \circ z + y \circ b - a \circ y$. Since $\phi \in Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$, it follows that $y \circ b - a \circ y = 0$ and so $\phi(m) = x \circ m - m \circ z = \rho_{x,z}(m)$ for all $m \in \mathcal{M}$. Thus $\phi \in ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$, as claimed. $\square$

The next theorem is a direct analogue of [7], Thm. 2.8.

3.12. Theorem. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach algebras and $\mathcal{M}$ be a unital Banach $\mathcal{A}, \mathcal{B}$-module. Let $\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{bmatrix}$ be the corresponding triangular Banach algebra. If n is a positive integer and both $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly amenable, then*

$$H^1(\mathcal{T}, \mathcal{T}^{(2n)}) \simeq Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) / ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}).$$

Proof. Let $\Theta : Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}) \to H^1(\mathcal{T}, \mathcal{T}^{(2n)})$ be defined by $\Theta(\phi) = \overline{\Delta_\phi}$, where $\overline{\Delta_\phi}$ denotes the equivalence class of Δ_ϕ in $H^1(\mathcal{T}, \mathcal{T}^{(2n)})$. Then it is relatively simple to verify that Θ is linear.

To see that Θ is surjective, let $\delta : \mathcal{T} \to \mathcal{T}^{(2n)}$ be a continuous derivation. By Proposition 3.9, we can find derivations $\delta_1 : \mathcal{A} \to \mathcal{A}^{(2n)}$, $\delta_4 : \mathcal{B} \to \mathcal{B}^{(2n)}$, a linear map $\rho : \mathcal{M} \to \mathcal{M}^{(2n)}$ and an element $\gamma_\delta \in \mathcal{M}^{(2n)}$ such that

$$\delta \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{bmatrix} \delta_1(a) & a \circ \gamma_\delta - \gamma_\delta \circ b + \rho(m) \\ & \delta_4(b) \end{bmatrix}.$$

Our assumption that $\mathcal{A}$ and $\mathcal{B}$ are $(2n)$-weakly amenable implies the existence of $x \in \mathcal{A}^{(2n)}$, $z \in \mathcal{B}^{(2n)}$ such that $\delta_1 = \delta_x$ and $\delta_4 = \delta_z$. Define $\delta_0 : \mathcal{T} \to \mathcal{T}^{(2n)}$ via

$$\delta_0 \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{bmatrix} \delta_x(a) & \rho_{x,z}(m) + a \circ \gamma_\delta - \gamma_\delta \circ b \\ & \delta_z(b) \end{bmatrix}.$$

Then $\delta_0 = \delta_{\begin{bmatrix} x & -\gamma_\delta \\ & z \end{bmatrix}}$ is inner, and hence $\delta_1 = \delta - \delta_0$ satisfies $\overline{\delta_1} = \overline{\delta}$ in $H^1(\mathcal{T}, \mathcal{T}^{(2n)})$. Furthermore, $\delta_1 \begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{bmatrix} 0 & \rho_1(m) \\ & 0 \end{bmatrix}$, where $\rho_1 = \rho - \rho_{x,z}$. Since $\rho, \rho_{x,z}$ belong to $Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$, so does ρ_1. By Proposition 3.9 (ii), $\Theta(\rho_1) = \overline{\Delta_{\rho_1}} = \overline{\delta_1} = \overline{\delta}$, showing that Θ is indeed onto.

Next, $\phi \in \ker \Theta$ precisely when $\Theta(\phi)$ is inner. Again, by Lemma 3.11, this is equivalent to the assertion that $\phi \in ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})$. It then follows that

$$\begin{aligned} \operatorname{ran} \Theta = H^1(\mathcal{T}, \mathcal{T}^{(2n)}) &\simeq Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})/\ker \Theta \\ &= Hom_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)})/ZR_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{M}^{(2n)}). \quad \square \end{aligned}$$

4. Examples

4.1. **Example One: $\mathcal{T}_2 \otimes \mathcal{A}$, where $\mathcal{A}$ is a unital C^*-algebra.** Suppose $\mathcal{B}$ is a unital Banach algebra and consider $\mathcal{T}_\mathcal{B} = \mathcal{T}_2 \otimes \mathcal{B}$. By Theorem 3.4, $H^1(\mathcal{T}_\mathcal{B}, \mathcal{T}_\mathcal{B}^{(2n-1)}) \simeq H^1(\mathcal{B}, \mathcal{B}^{(2n-1)}) \oplus H^1(\mathcal{B}, \mathcal{B}^{(2n-1)})$. In the case where $\mathcal{A}$ is a unital C^*-algebra, Theorem 3.1 of [5] states that $\mathcal{A}$ is permanently weakly amenable, and hence if we set $\mathcal{T} = \mathcal{T}_2 \otimes \mathcal{A}$, then $H^1(\mathcal{T}, \mathcal{T}^{(2n-1)}) \simeq 0 \oplus 0 = 0$. That is, $\mathcal{T}$ is $(2n-1)$-weakly amenable for all positive integers n.

As for $(2n)$-weak amenability of $\mathcal{T}$, we first recall the following result.

4.2. Proposition ([5], Cor. 1.12)**.** *Let $\mathcal{B}$ be a Banach algebra such that $\mathcal{B}^{(2n)}$ is Arens regular, and suppose that $H^1(\mathcal{B}^{(2n+2)}, \mathcal{B}^{(2n+2)}) = 0$ for each positive integer n. Then $\mathcal{B}$ is $(2n)$-weakly amenable for each $n \in \mathbb{N}$.*

It is well-known that every unital C^*-algebra $\mathcal{A}$ is Arens regular [4], and thus so is $\mathcal{A}^{(2n)}$, $n \geq 1$, since each $\mathcal{A}^{(2n)}$ is a unital von Neumann algebra. By Proposition 2.3, each $\mathcal{T}^{(2n)} = \mathcal{T}_2 \otimes \mathcal{A}^{(2n)}$ is Arens regular, and in order to apply Proposition 4.2 to $\mathcal{T}$, it suffices to show that

$$H^1(\mathcal{T}^{(2n+2)}, \mathcal{T}^{(2n+2)}) = 0$$

for each $n \in \mathbb{Z}^+$. We first require a result which is in fact the converse of [7], Proposition 3.3. That result in turn depends on simple algebraic lemma, the following whose proof is left to the reader.

4.3. Lemma. *Let V be a unital algebra. Then $Hom_V(V) \simeq Z(V) \simeq ZR_V(V)$, where $Hom_V(V)$ denotes the V-module maps from V into itself, $Z(V)$ denotes the centre of V, and $ZR_V = \{\rho_{x,z} : V \to V; x, z \in Z(V)\}$.*

4.4. Proposition. *Let $\mathcal{B}$ be a unital Banach algebra. Then $H^1(\mathcal{B}, \mathcal{B}) = 0$ if and only if $H^1(\mathcal{T}_2 \otimes \mathcal{B}, \mathcal{T}_2 \otimes \mathcal{B}) = 0$.*

Proof. If $H^1(\mathcal{T}_2 \otimes \mathcal{B}, \mathcal{T}_2 \otimes \mathcal{B}) = 0$, then $H^1(\mathcal{B}, \mathcal{B}) = 0$ by Proposition 3.3 of [7]. If $H^1(\mathcal{B}, \mathcal{B}) = 0$, then by Theorem 2.8 of the same paper,

$$\begin{aligned} H^1(\mathcal{T}_2 \otimes \mathcal{B}, \mathcal{T}_2 \otimes \mathcal{B}) &\simeq Hom_\mathcal{B}(\mathcal{B})/ZR_\mathcal{B}(\mathcal{B}) \\ &= 0 \end{aligned}$$

by Lemma 4.3 above. $\square$

Equipped with the above proposition, we can now complete our analysis of $H^1(\mathcal{T}^{(2n)}, \mathcal{T}^{(2n)})$. Indeed, since $\mathcal{A}^{(2n+2)}$ is a unital von Neumann algebra for each positive integer n, a theorem of Sakai [14] (see also Kadison [11] for an independent proof) asserts that each $H^1(\mathcal{A}^{(2n+2)}, \mathcal{A}^{(2n+2)}) = 0$. Proposition 4.4 then shows that $H^1(\mathcal{T}^{(2n+2)}, \mathcal{T}^{(2n+2)}) = 0$, which is, as pointed out earlier, sufficient to prove that $\mathcal{T}$ is $(2n)$-weakly amenable for all natural numbers n.

Summarizing:

4.5. Proposition. *If $\mathcal{A}$ is a unital C^*-algebra, then $\mathcal{T} = \mathcal{T}_2 \otimes \mathcal{A}$ is permanently weakly amenable.*

4.6. **Example Two: odd versus even weak amenability.** Our next example shows that permanent weak amenability is not guaranteed by the fact that the corner algebras are themselves permanently weakly amenable.

Let $\mathcal{H}$ be an infinite dimensional, complex, separable Hilbert space, let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$, and let $\mathcal{K}(\mathcal{H})$ denote the closed, two-sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$. We set $\mathcal{Q}(\mathcal{H})$ to denote the quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{Q}(\mathcal{H})$ denote the canonical quotient map. Consider the algebra $\mathcal{A} = \mathbb{C}I + \mathcal{K}(\mathcal{H})$, the unitization of $\mathcal{K}(\mathcal{H})$, and let

$$\mathcal{T} = \begin{bmatrix} \mathcal{A} & \mathcal{B}(\mathcal{H}) \\ & \mathcal{A} \end{bmatrix}.$$

As we saw in the previous example, since $\mathcal{A}$ is a unital C^*-algebra, [5], Theorem 3.1 implies that $\mathcal{A}$ is permanently weakly amenable. Applying Theorem 3.7 , we obtain

$$\mathcal{T} \text{ is } (2n-1)\text{-weakly amenable for all } n \geq 1.$$

As we shall now see, $\mathcal{T}$ is not 2-weakly amenable (and hence $\mathcal{T}$ is not $(2n)$-weakly amenable for any $n \geq 1$, by [5], Proposition 1.2). The proof is an application of Theorem 3.12. Namely,

$$H^1(\mathcal{T}, \mathcal{T}^{(2)}) \simeq Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)})/ZR_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)}).$$

Before we begin, let us briefly describe $\mathcal{B}(\mathcal{H})^{(2)}$, as well as the action of $\mathcal{B}(\mathcal{H})$ and $\mathcal{A}$ upon $\mathcal{B}(\mathcal{H})^{(2)}$. Recall that a closed subspace $\mathfrak{Y}$ of a Banach space $\mathfrak{X}$ is called an *M-ideal* if there is a linear projection $\eta : \mathfrak{X}^* \to \mathfrak{Y}^\perp$, where $\mathfrak{Y}^\perp = \{\psi \in \mathfrak{X}^* : \psi|_{\mathfrak{Y}} = 0\}$, and if for each $\phi \in \mathfrak{X}^*$ we have $\|\phi\| = \|\eta(\phi)\| + \|\phi - \eta(\phi)\|$. We refer the reader to [6], Chapter 11, for more information regarding M-ideals. For our purposes, we shall only require the following facts:

(a) If $\mathfrak{Y}$ is an M-ideal of a Banach space $\mathfrak{X}$, then every $y^* \in \mathfrak{Y}^*$ has a *unique* Hahn-Banach extension to $\mathfrak{X}^*$. This allows us to view $\mathfrak{Y}^*$ as a subspace of $\mathfrak{X}^*$, and with respect to this embedding, $\mathfrak{X}^* = \mathfrak{Y}^* \oplus_1 \mathfrak{Y}^\perp$ (i.e. the ℓ^1-direct sum).

(b) A subspace $\mathcal{K}$ of a C^*-algebra $\mathcal{B}$ is an M-ideal if and only if $\mathcal{K}$ is a closed, two-sided ideal of $\mathcal{B}$.

(c) If $\mathcal{B}$ is a C^*-algebra and $\mathcal{K}$ is a closed, two-sided ideal of $\mathcal{B}$, then $\mathcal{B}^* \simeq \mathcal{K}^* \oplus_1 \mathcal{K}^\perp$. Hence $\mathcal{B}^{(2)} \simeq \mathcal{K}^{(2)} \oplus_\infty (\mathcal{K}^\perp)^*$, and both $\mathcal{K}^{(2)}$ and $(\mathcal{K}^\perp)^*$ are ideals in $\mathcal{B}^{(2)}$.

Before applying this to our example, recall also that

(i) $\mathcal{K}(\mathcal{H})^*$ is isometrically isomorphic to $\mathcal{C}_1(\mathcal{H})$, the *trace class* operators on $\mathcal{H}$, via the map

$$\begin{array}{rccc} \nu: & \mathcal{C}_1(\mathcal{H}) & \to & \mathcal{K}(\mathcal{H})^*, \\ & C & \mapsto & \phi_C, \end{array}$$

where $\phi_C(K) = \mathrm{tr}(CK)$, $K \in \mathcal{K}(\mathcal{H})$. Here, $\mathrm{tr}(\cdot)$ represents the trace functional on $\mathcal{C}_1(\mathcal{H})$.

(ii) $\mathcal{K}(\mathcal{H})^{(2)} \simeq \mathcal{C}_1(\mathcal{H})^*$ is isometrically isomorphic to $\mathcal{B}(\mathcal{H})$ via a similar map:

$$\begin{array}{rccc} \theta: & \mathcal{B}(\mathcal{H}) & \to & \mathcal{C}_1(\mathcal{H})^*, \\ & X & \mapsto & \phi_X, \end{array}$$

where $\phi_X(C) = \mathrm{tr}(XC)$ for each $C \in \mathcal{C}_1(\mathcal{H})$.

By (a) and (b) above, we have

$$\begin{array}{rcl} \mathcal{B}(\mathcal{H})^* & \simeq & \mathcal{K}(\mathcal{H})^* \oplus_1 \mathcal{K}(\mathcal{H})^\perp \\ & \simeq & \mathcal{C}_1(\mathcal{H}) \oplus_1 (\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}))^* \\ & \simeq & \mathcal{C}_1(\mathcal{H}) \oplus_1 \mathcal{Q}(\mathcal{H})^*, \end{array}$$

and

$$\begin{array}{rcl} \mathcal{B}(\mathcal{H})^{(2)} & \simeq & \mathcal{C}_1(\mathcal{H})^* \oplus_\infty \mathcal{Q}(\mathcal{H})^{(2)} \\ & \simeq & \mathcal{B}(\mathcal{H}) \oplus_\infty \mathcal{Q}(\mathcal{H})^{(2)}. \end{array}$$

Moreover, $\mathcal{B}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})^{(2)}$ are ideals in $\mathcal{B}(\mathcal{H})^{(2)}$.

Consider the action of $\mathcal{K}(\mathcal{H})$ on $\mathcal{B}(\mathcal{H})^*$. Given $K \in \mathcal{K}(\mathcal{H})$ and $\phi_C \in \mathcal{C}_1(\mathcal{H})$, we see that $(K \circ \phi_C)(L) = \phi_C(L\,K) = \mathrm{tr}(C\,L\,K) = \mathrm{tr}(K\,C\,L) = \phi_{K\,C}(L)$ for all $L \in \mathcal{K}(\mathcal{H})$, and hence $(K \circ \phi_C) = \phi_{K\,C}$. Similarly, $(\phi_C \circ K) = \phi_{C\,K}$. As for the action of K on $\sigma \in \mathcal{Q}(\mathcal{H})^*$, $(K \circ \sigma)(a) = \sigma(K \circ a) = \sigma(\pi(K)\cdot a) = \sigma(0\cdot a) = \sigma(0) = 0$ for all $a \in \mathcal{Q}(\mathcal{H})$, and so $K \circ \sigma = 0$. Likewise, $\sigma \circ K = 0$.

Considering next the action of $\mathcal{K}(\mathcal{H})$ on $\mathcal{B}(\mathcal{H})^{(2)}$, we find that if $\gamma_X \in \mathcal{C}_1(\mathcal{H})^* \simeq \mathcal{B}(\mathcal{H})$, then calculations similar to those above show that $K \circ \gamma_X = \gamma_{K\,X}$, $\gamma_X \circ K = \gamma_{X\,K}$ for all $K \in \mathcal{K}(\mathcal{H})$. As such, this action is equivalent to usual operator multiplication. From above, if $\beta \in \mathcal{Q}(\mathcal{H})^{(2)}$, then $(K \circ \beta)(\sigma) = \beta(\sigma \circ K) = \beta(0) = 0 = \beta(K \circ \sigma) = (\beta \circ K)(\sigma)$. Thus $\beta \circ K = K \circ \beta = 0$ for all $K \in \mathcal{K}(\mathcal{H})$ and $\beta \in \mathcal{Q}(\mathcal{H})^{(2)}$, so that $\mathcal{K}(\mathcal{H})$ acts trivially on $\mathcal{Q}(\mathcal{H})^{(2)}$.

With this background, we can (finally!) compute $Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)})$.

Given $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})^{(2)} \simeq \mathcal{B}(\mathcal{H}) \oplus \mathcal{Q}(\mathcal{H})^{(2)}$, we may decompose Φ as (Φ_1, Φ_2), where

$$\Phi_1(X) = P_1(X) \quad \text{and} \qquad \begin{array}{rccc} P_1: & \mathcal{B}(\mathcal{H})^{(2)} & \to & \mathcal{B}(\mathcal{H}), \\ & (Z,\beta) & \mapsto & Z, \end{array}$$

while

$$\Phi_2(X) = P_2(X) \quad \text{and} \qquad \begin{array}{rccc} P_2: & \mathcal{B}(\mathcal{H})^{(2)} & \to & \mathcal{Q}(\mathcal{H})^{(2)}, \\ & (Z,\beta) & \mapsto & \beta. \end{array}$$

Since both $\mathcal{B}(\mathcal{H})$ and $\mathcal{Q}(\mathcal{H})^{(2)}$ are ideals in $\mathcal{B}(\mathcal{H})^{(2)}$, it follows that

$$\Phi \in Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)})$$

if and only if

$$\Phi_1 \in Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})) \quad \text{and} \quad \Phi_2 \in Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{Q}(\mathcal{H})^{(2)}).$$

That is,

$$Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})^{(2)}) \simeq Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})) \oplus Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)}).$$

But $Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})) = Hom_{\mathcal{K}(\mathcal{H}),\mathcal{K}(\mathcal{H})}(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})) \simeq \mathbb{C}$, as is readily verified.

We claim that $Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)}) \simeq \mathcal{B}(\mathcal{Q}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)})$, the bounded linear operators from the Calkin algebra into its second dual. To see this, suppose that $\phi \in \mathcal{B}(\mathcal{Q}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)})$, and let $\Phi = \phi \circ \pi \in \mathcal{B}(\mathcal{B}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)})$, the space of bounded linear operators from $\mathcal{B}(\mathcal{H})$ into $\mathcal{Q}(\mathcal{H})^{(2)}$. If $\lambda \in \mathbb{C}$, $K \in \mathcal{K}(\mathcal{H})$, and $X \in \mathcal{B}(\mathcal{H})$, then

$$\begin{aligned}\Phi((\lambda I + K)X) &= \phi(\pi((\lambda I + K)X)) \\ &= \lambda\phi(\pi(X)) \\ &= \lambda\Phi(X) \\ &= \Phi(X(\lambda I + K)).\end{aligned}$$

In other words, $\Phi \in Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)})$.

It is not hard to verify that the map $\Theta : \phi \mapsto \Phi = \phi \circ \pi$ is linear and injective. It remains to prove that it is surjective. To that end, suppose that $\Psi \in Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)})$, and let $K \in \mathcal{K}(\mathcal{H})$. Then $\Psi(K) = \Psi(K \cdot 1) = K \circ \Psi(1) = 0$, since the action of $\mathcal{K}(\mathcal{H})$ on $\mathcal{Q}(\mathcal{H})^{(2)}$ is trivial. This means that the map

$$\begin{aligned}\psi: \quad \mathcal{Q}(\mathcal{H}) &\rightarrow \mathcal{Q}(\mathcal{H})^{(2)}, \\ \pi(X) &\mapsto \Psi(X),\end{aligned}$$

is well-defined and bounded. (Indeed, if $\|\pi(X)\| = 1$, we can choose $K \in \mathcal{K}(\mathcal{H})$ such that $\|X + K\| = 1$, and then $\|\psi(\pi(X))\| = \|\Psi(X)\| = \|\Psi(X + K)\| \leq \|\Psi\| \, \|X + K\| = \|\Psi\|$.) Moreover, it is clear that $\Theta(\psi) = \Psi$, and thus Θ is a linear isomorphism.

So far we have shown that

$$Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H})^{(2)}) \simeq \mathbb{C} \oplus \mathcal{B}(\mathcal{Q}(\mathcal{H}),\mathcal{Q}(\mathcal{H})^{(2)}).$$

Next we must consider $ZR_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{H}))$. Borrowing from our above analysis, $\mathcal{K}(\mathcal{H})$ is an ideal in the C^*-algebra $\mathcal{A}$, and so

$$\begin{aligned}\mathcal{A}^{(2)} &= \mathcal{K}(\mathcal{H})^{(2)} \oplus (\mathcal{A}/\mathcal{K}(\mathcal{H}))^{(2)} \\ &= \mathcal{B}(\mathcal{H}) \oplus \mathbb{C},\end{aligned}$$

and both $\mathcal{B}(\mathcal{H})$ and $\mathbb{C}$ are ideals of $\mathcal{A}^{(2)}$. The action of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$ is again just usual operator multiplication. As for the action of $\mathcal{A}$ on $\mathbb{C}$, as before it must annihilate $\mathcal{K}(\mathcal{H})$, and so we have

$$(\lambda I + K) \circ \alpha = \lambda\,\alpha = \alpha \circ (\lambda I + K)$$

for all λ, $\alpha \in \mathbb{C}$ and $K \in \mathcal{K}(\mathcal{H})$.

This tells us that $(X, \alpha) \in Z_{\mathcal{A}}(\mathcal{A}^{(2)})$ implies that $((\lambda I + K)X, \lambda\,\alpha) = (\lambda I + K) \circ (X, \alpha) = (X, \alpha) \circ (\lambda I + K) = (X(\lambda I + K), \alpha\,\lambda)$ for all $\lambda \in \mathbb{C}$ and $K \in \mathcal{K}(\mathcal{H})$. Thus X lies in the commutant of $\mathcal{K}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$, which is $\mathbb{C}I$, since $\mathcal{K}(\mathcal{H})$ is an irreducible C^*-algebra [12]. The converse is trivial; namely, if $\kappa, \alpha \in \mathbb{C}$, then $(\kappa I, \alpha) \in Z_{\mathcal{A}}(\mathcal{A}^{(2)})$.

Then $ZR_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)}) = \{\tau_{A,B} : A,\, B \in Z_{\mathcal{A}}(\mathcal{A}^{(2)})\}$, and

$$\begin{aligned}\tau_{A,B}(X) &= A \circ X - X \circ B \\ &= (\kappa_A I, \alpha_A) \circ X - X \circ (\kappa_B I, \alpha_B) \\ &= (\kappa_A X, \alpha_A \pi(X)) - (\kappa_B X, \alpha_B \pi(X)) \\ &= ((\kappa_A - \kappa_B) I, (\alpha_A - \alpha_B)) \circ X.\end{aligned}$$

Thus $ZR_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)}) \simeq \mathbb{C} \oplus \mathbb{C}$ (with elements of the former set being identified with scalar multiplication operators). Finally,

$$\begin{aligned}Hom_{\mathcal{A},\mathcal{A}}(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^{(2)}) &\simeq \frac{\mathbb{C} \oplus \mathcal{B}(\mathcal{Q}(\mathcal{H}), \mathcal{Q}(\mathcal{H})^{(2)})}{\mathbb{C} \oplus \mathbb{C}} \\ &\simeq \mathcal{B}(\mathcal{Q}(\mathcal{H}), \mathcal{Q}(\mathcal{H})^{(2)})/\mathbb{C} \neq 0.\end{aligned}$$

Thus, by Theorem 3.12, $\mathcal{T}$ is not 2-weakly amenable.

4.7. **Finite dimensional nest algebras are permanently weakly amenable.** Recall that a *nest* on a Hilbert space $\mathcal{H}$ is a chain $\mathcal{N}$ of closed subspaces of $\mathcal{H}$ which contains $\{0\}$ and $\mathcal{H}$, and which is closed under the operations of taking intersections and closed spans. The corresponding *nest algebra* $\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq N \text{ for all } N \in \mathcal{N}\}$.

Our main result here is the following:

4.8. Theorem. *Let $\mathcal{T}(\mathcal{N})$ be a nest algebra acting on a finite dimensional, complex Hilbert space $\mathcal{H}$. Then $\mathcal{T}(\mathcal{N})$ is permanently weakly amenable.*

Proof. Since $\mathcal{T}(\mathcal{N})$ is finite dimensional, it follows that $\mathcal{T}(\mathcal{N})^{(2n)}$ is isomorphic to $\mathcal{T}(\mathcal{N})$ as a Banach algebra for each $n \geq 1$. But then

$$H^1(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})^{(2n)}) \simeq H^1(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})) = 0$$

for all $n \geq 1$ ([3], or [14], when the nest is trivial and $\mathcal{T}(\mathcal{N}) \simeq \mathcal{B}(\mathcal{H})$).

As for odd weak amenability, the proof works by induction on the dimension m of the space $\mathcal{H}$ upon which $\mathcal{T}(\mathcal{N})$ acts. Of course, when $m = 1$, $\mathcal{T}(\mathcal{N}) \simeq \mathbb{C}$ is an abelian C^*-algebra; hence it is amenable (and therefore permanently weakly amenable). Suppose the result holds for all nest algebras acting upon Hilbert spaces of dimension less than k. Let $\mathcal{T}(\mathcal{N})$ be a nest algebra acting on a space of dimension k. Then, either the nest $\mathcal{N}$ is trivial, in which case $\mathcal{T}(\mathcal{N}) \simeq \mathcal{B}(\mathcal{H})$ is a C^*-algebra, hence is permanently weakly amenable by [5], Thm. 2.1, or there exists an element $\{0\} \neq N \neq \mathcal{H}$ in $\mathcal{N}$ and we may decompose $\mathcal{T}(\mathcal{N})$ as

$$\mathcal{T}(\mathcal{N}) = \begin{bmatrix} \mathcal{A} & \mathcal{M} \\ 0 & \mathcal{B} \end{bmatrix} \begin{matrix} N \\ N^\perp \end{matrix}$$

where $\mathcal{A}$ is a nest algebra acting upon N, $\mathcal{B}$ is a nest algebra acting upon $N^\perp$, and $\mathcal{M}$ is the space of all operators taking $N^\perp$ into N. By our induction hypothesis, $\mathcal{A}$ and $\mathcal{B}$ are permanently weakly amenable. Applying Theorem 3.7 above, we conclude that $\mathcal{T}(\mathcal{N})$ is $(2n-1)$-weakly amenable for all $n \geq 1$.

Combining these two results yields our theorem. □

Of course, the above theorem begs the question which we have been unable to resolve as of this time, namely: are all nest algebras permanently weakly amenable?

References

[1] Bade, W.G., Curtis, P.C. and Dales, H.G., *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. **55** (1987), 359-377. MR **88f:**46098

[2] Bade, W.G., Dales, H.G. and Lykova, Z.A., *Algebraic and strong splittings of extensions of Banach algebras*, Mem. Amer. Math. Soc., 137 (1999), No. 656. MR **99g:**46059

[3] Christensen, E., *Derivations of nest algebras*, Math. Ann., **229** (1977), 155-161. MR **56:**6420

[4] Civin, P. and Yood, B., *The second conjugate space of a Banach algebra as an algebra*, Pacific J. Math., **11** (1961), 847-870. MR **26:**622

[5] Dales, H.G., Ghahramani, F. and Grønbæk, N., *Derivations into iterated duals of Banach algebras*, Studia Math., **128** (1998), 19–54. MR **99g:**46064

[6] Davidson, K.R., *Nest Algebras*, Pitman Research Notes in Mathematics, no. 191, Longman Scientific and Technical, London, 1988. MR **90f:**47062

[7] Forrest, B.E. and Marcoux, L.W., *Derivations of triangular Banach algebras*, Indiana Univ. Math. J. **45** (1996), 441–462. MR **97m:**46079

[8] Gilfeather, F. and Smith, R.R., *Cohomology for operator algebras: joins*, Amer. J. Math. **116** (1994), 541–561. MR **95d:**46077

[9] Haagerup, U., *All nuclear C^*-algebras are amenable*, Invent. Math. **74** (1983), 305–319. MR **85g:**46074

[10] Johnson, B.E., *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., 127 (1992). MR **51:**11130

[11] Kadison, R.V., *Derivations of operator algebras*, Ann. Math., **83** (1966), 280-293. MR **33:**1747

[12] Murphy, G.J., *C^*-Algebras and Operator Theory*, Academic Press, San Diego, 1990. MR **91m:**46084

[13] Palmer, T.W., *Banach algebras and the general theory of *-algebras, Vol. 1, Algebras and Banach Algebras*, Cambridge Univ. Press, 1994. MR **95c:**46002

[14] Sakai, S., *Derivations of W^*-algebras*, Ann. Math., **83** (1966), 273-279. MR **33:**1748

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: `beforres@math.uwaterloo.ca`

Department of Mathematical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Current address: Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: `L.Marcoux@ualberta.ca`